METRIC AND TOPOLOGICAL SPACES: EXAM 2023/24

A. V. KISELEV

Problem 1 (20%). The discrete metric d_0 on $\mathbb{R}$ attains exactly two distinct values. Can a metric ϱ on $\mathbb{R}$ attain exactly four distinct values? (If not, prove; if yes, give example)

Problem 2 (15 + 10%). (*a*) If $A \subsetneq \mathcal{X}$ is a complete subset of a metric space $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$, then A is closed in $\mathcal{X}$. (prove)
(*b*) Give an example $A \subsetneq \mathcal{X}$ of (*a*) when $\mathcal{X}$ itself is not complete.

Problem 3 (15%). Give an example of a sequence of open connected subsets $L_n \subsetneq \mathbb{E}^3$ in space such that $L_n \supseteq L_{n+1}$ for all $n \in \mathbb{N}$ but the intersection $\bigcap_{n=1}^{+\infty} L_n$ is not connected.
(NB: $\varnothing$ is connected)

Problem 4 (10 + 10%). (*a*) Suppose $V_n \neq \varnothing$, $n \in \mathbb{N}$, is a closed subset of sequentially compact space $\mathcal{X}$, and $V_n \supseteq V_{n+1}$. Prove that $\bigcap_{n=1}^{+\infty} V_n \neq \varnothing$.
(*b*) Is this intersection always non-empty if $\mathcal{X}$ is not sequentially compact? (state and prove, e.g., by counterexample)

Problem 5 (20%). Find a solution $x(s)$ of the integral equation,

$$x(s) = \frac{1}{2}\int_0^1 x(t)\,\mathrm{d}t + \exp(s) - \tfrac{1}{2}(\exp(1) - 1),$$

by consecutive approximations starting from $x_0(s) = 0$.
Please verify by direct substitution that it satisfies the equation!

Date: November 6, 2023. Good luck & take care!